One Rude Passenger's Contagion

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I was recently posed the following problem:

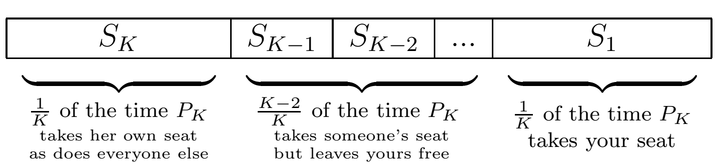
Suppose there is an airplane with K seats. The 1st passenger is quite rude and, disregarding her ticket, picks any seat uniformly at random. The following passengers try to take their own seat if it's available, otherwise, being too polite to speak up, they pick a free seat uniformly at random. If you are the last passenger to board, what is the probability of your assigned seat being unoccupied when you get there?

It's worthwhile to think about this yourself before reading the solution below.

We will track the state of seat i with:

$$S_i = \begin{cases} 0 & \text{if seat } i \text{ is free} \\ x & \text{if seat } i \text{ is occupied by passenger } x \end{cases}$$

To ease notation, let passenger P_K with assigned seat K be the first to board. This is the rude 1st passenger. Passenger P_{K-1} boards next, then P_{K-2}, etc. You are P_1 and board last. Let's see what options P_K has:



The 1st case is simple. If P_K takes her own seat, then P_{K-1}'s seat is available when she boards as is every following passenger's seat. The last case is also easy: P_K took your seat - game over - it won't be available when you get there.

The middle case is more interesting. It is easy to get lost thinking about possible interactions between passengers stealing each other's seats. The reality is much simpler. Only one passenger's seat is incorrectly occupied at any given time. Examine the following situation in which $S_{K-3} = K$:

			K	

It is clear that first P_{K-1} and then P_{K-2} will both be able to take their seats in order:

	$K-1$	$K-2$	K	

Once P_{K-3} gets on the plane her seat is taken. The contagion has spread and P_{K-3} must now pick an available seat uniformly at random. Notice, however, that $S_K = 0$ and if that seat is picked every following passenger will once again be guaranteed their assigned seating. P_{K-3} is in exactly P_K's situation but on a subproblem with fewer seats.

With that, let $f(k)$ be the probability that your seat is free when you board last on a plane with k seats:

$$f(k) = \begin{cases} \frac{1}{k} + \frac{k-2}{k} f(k-1) + \frac{1}{k} 0 & \text{if } S_k > 0 \text{ or } k = K \\ f(k-1) & \text{if } S_k = 0 \end{cases}$$

Note that $f(2) = \frac{1}{2}$ because $P_{K=2}$ only has a choice of taking her own seat or yours. This forms the base case of an induction proof.

Theorem 1. *The probability of finding your seat unoccupied is $\frac{1}{2}$ regardless of the size of the plane — ie. $f(k) = \frac{1}{2}$ for $k \geq 2$.*

Proof. Suppose that $f(k-1) = \frac{1}{2}$ for some $k-1$.

$$f(k) = \begin{cases} \dfrac{1}{k} + \dfrac{k-2}{k} f(k-1) + \dfrac{1}{k} 0 & \text{if } S_k > 0 \\ f(k-1) & \text{if } S_k = 0 \end{cases} \tag{1}$$

$$= \begin{cases} \dfrac{1}{k} + \dfrac{k-2}{k} \dfrac{1}{2} & \text{if } S_k > 0 \\ \dfrac{1}{2} & \text{if } S_k = 0 \end{cases} \tag{2}$$

$$= \begin{cases} \dfrac{1}{2} & \text{if } S_k > 0 \\ \dfrac{1}{2} & \text{if } S_k = 0 \end{cases} \tag{3}$$

$$= \frac{1}{2} \tag{4}$$

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